Mazur's theorem tells us that the points of finite order on an elliptic curve over the rationals has to have a particular structure. In particular, if E_{tors} is the subgroup of $E(Q)$ of points of finite order then E_{tors} has to have one of the following forms:

1. Z_n, a cyclic group of order n where $1 \leq n \leq 10$ or $n = 12$
2. Z_2 x Z_{2n}, the direct product of a cyclic group of order 2 with one of order $2n$ for $1 \leq n \leq 4$

Here are examples of elliptic curves for each one of these possibilities (from Exercise 8.12 on p. 238 of Silverman's *The Arithmetic of Elliptic Curves*).

E/Q	E_{tors}
$y^2 = x^3 - 2$	$Z_1 = \{O\}$
$y^2 = x^3 + 8$	$Z_2 = \langle(-2,0)\rangle$
$y^2 = x^3 + 4$	$Z_3 = \langle(0,2)\rangle$
$y^2 = x^3 + 4x$	$Z_4 = \langle(2,4)\rangle$
$y^2 - y = x^3 + x^2$	$Z_5 = \langle(0,1)\rangle$
$y^2 = x^3 + 1$	$Z_6 = \langle(2,3)\rangle$
$y^2 = x^3 - 43x + 166$	$Z_7 = \langle(3,8)\rangle$
$y^2 + 7xy = x^3 + 16x$	$Z_8 = \langle(-2,10)\rangle$
$y^2 + xy + y = x^3 – x^2 - 14x + 29$	$Z_9 = \langle(3,1)\rangle$
$y^2 + xy = x^3 - 45x + 81$	$Z_{10} = \langle(0,9)\rangle$
$y^2 + 43xy – 210 y = x^3 - 210x$	$Z_{12} = \langle(0,210)\rangle$
$y^2 = x^3 - 4x$	Z_2 x $Z_2 = \langle(2,0), (0,0)\rangle$
$y^2 + xy – 5y = x^3 - 5x^2$	Z_2 x $Z_4 = \langle(10,20), (1,2)\rangle$
$y^2 + 5xy – 6y = x^3 - 3x^2$	Z_2 x $Z_6 = \langle(-3,18), (2,-2)\rangle$
$y^2 + 17xy – 120y = x^3 - 60x$	Z_2 x $Z_8 = \langle(30,-90), (-40,400)\rangle$

I was curious what each of these curves looked like, so I decided to graph both the curves and the points of E_{tors}. Some of the cases were interesting. Others were not.

One interesting fact was that some of the torsion points didn't have integer coordinates. That's not actually forbidden by the Nagell-Lutz theorem, which is this:

> Let $y^2 = x^3 + ax + b$ be an elliptic curve over the rationals with integer coefficients and let $D = 4\,a^3 + 27\,b^2$. Then if $P = (x_P, y_P)$ is a rational point of finite order then P has integer coordinates and either $y_P = 0$ *or* $y_P^2 | D$.

So if we have an elliptic curve that's in Weierstrass normal form then the Nagell-Lutz theorem doesn't tell us anything.

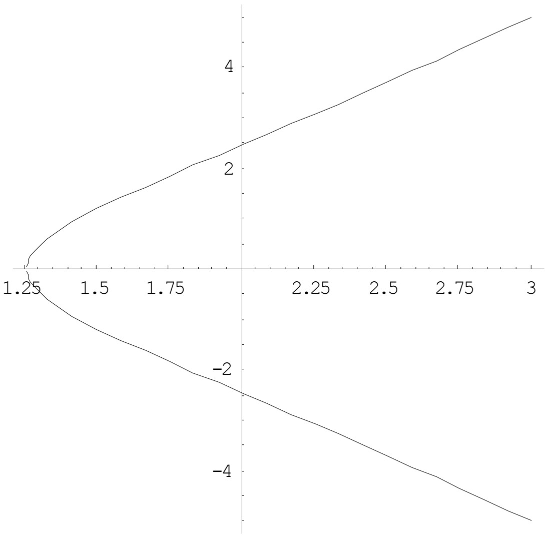
In any case, here's what I found.

Z_1

Here's the case where $y^2 = x^3 - 2$.

There are no finite torsion points, so $E_{\text{tors}} = Z_1 = \{O\}$. It's probably not one of the more interesting ones.

Here's what this looks like:

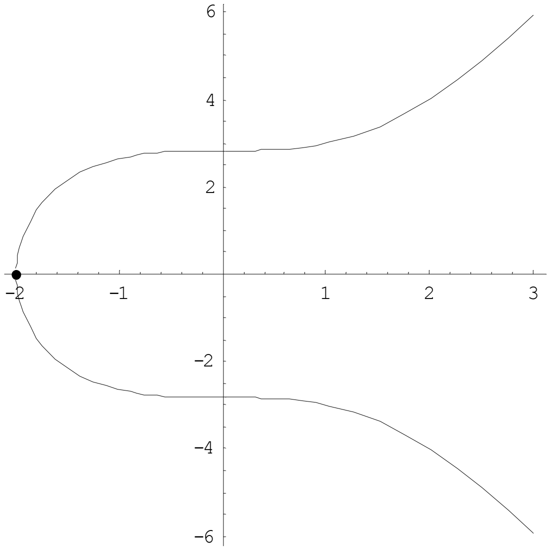


Z_2

Here's the case where $y^2 = x^3 + 8$.

Here we have that $E_{\text{tors}} = Z_2 = \langle(-2,0)\rangle = \{(-2,0), O\}$.

Here's what this looks like:

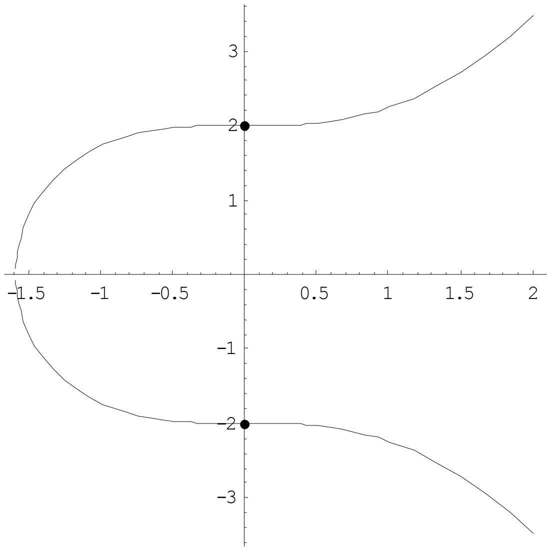


Z_3

Here's the case where $y^2 = x^3 + 4$.

Here we have that $E_{\text{tors}} = Z_3 = \langle(0,2)\rangle = \{(0,2), (0,-2), O\}$.

Here's what it looks like:

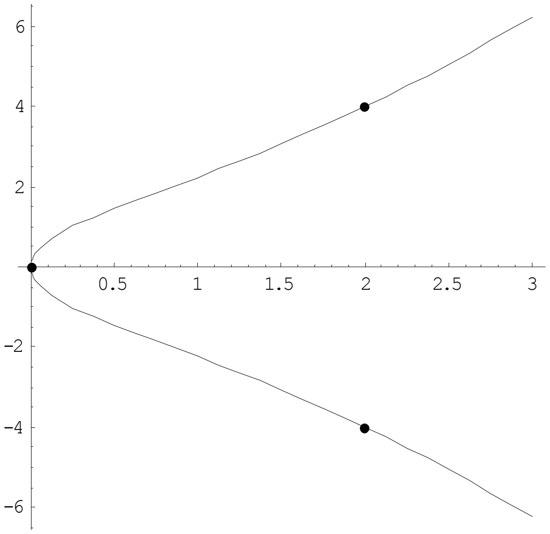


Z_4

Here's the case where $y^2 = x^3 + 4x$.

Here we have that $E_{\text{tors}} = Z_4 = \langle(2,4)\rangle = \{(2,4), (0,0), (2,-4), O\}$.

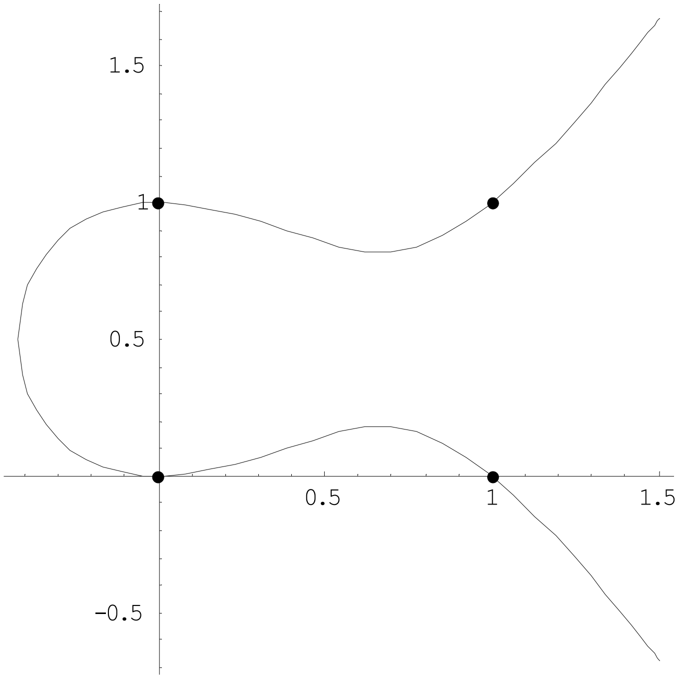
Here's what it looks like:



$\boldsymbol{Z_5}$

Here's the case where $y^2 - y = x^3 - x^2$.

Here we have that $E_{\text{tors}} = Z_5 = \langle(0,1)\rangle = \{(0,1), (1,0), (1,1), (0,0), O\}$.

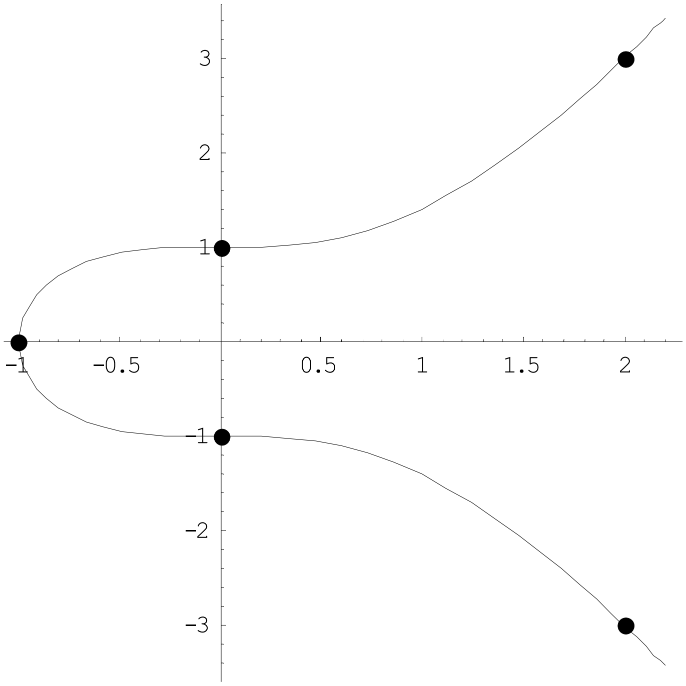


Z_6

Here's the case where $y^2 = x^3 + 1$.

Here we have that $E_{\text{tors}} = Z_6 = \langle(2,3)\rangle = \{(2,3), (0,1), (-1,0), (0,-1), (2,-3), O\}$.

Here's what it looks like:

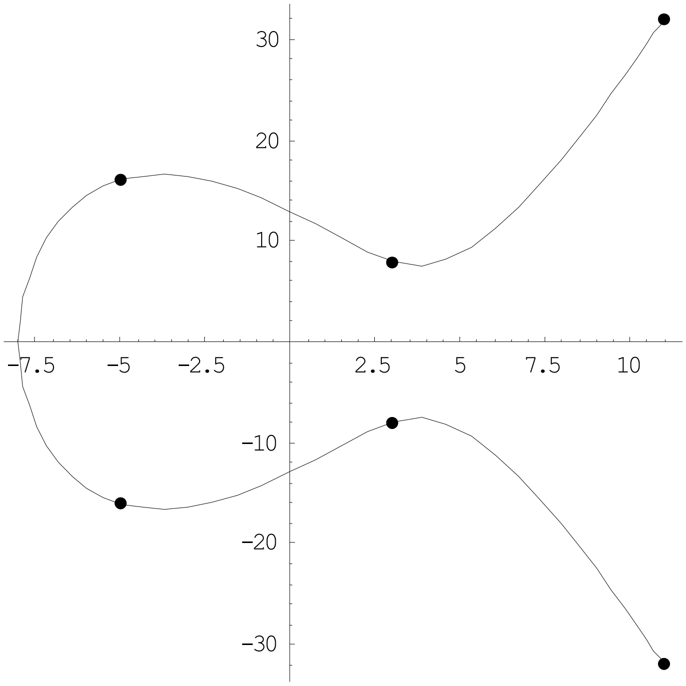


Z_7

Here's the case where $y^2 = x^3 - 43x + 166$.

Here we have that $E_{\text{tors}} = Z_7 = \langle(3,8)\rangle = \{(3,8), (-5,-16), (11,-32), (11,32), (-5,16), (3,-8), O\}$.

Here's what it looks like:

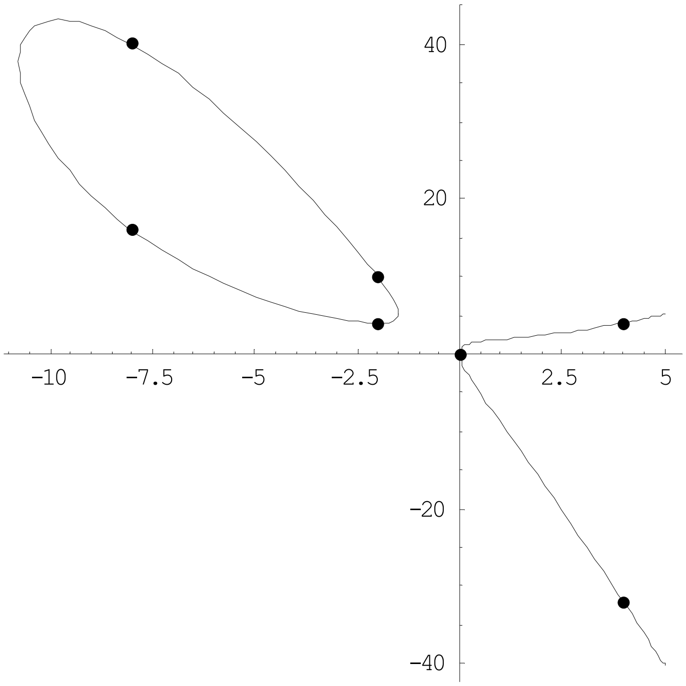


$\boldsymbol{Z_8}$

Here's the case where $y^2 + 7xy = x^3 + 6x$.

Here we have that $E_{\text{tors}} = Z_8 = \langle(-2,10)\rangle = \{(-2,10), (4,4), (-8,40), (0,0), (-8,16), (4,-32), (-2,4), O\}$.

Here's what it looks like:

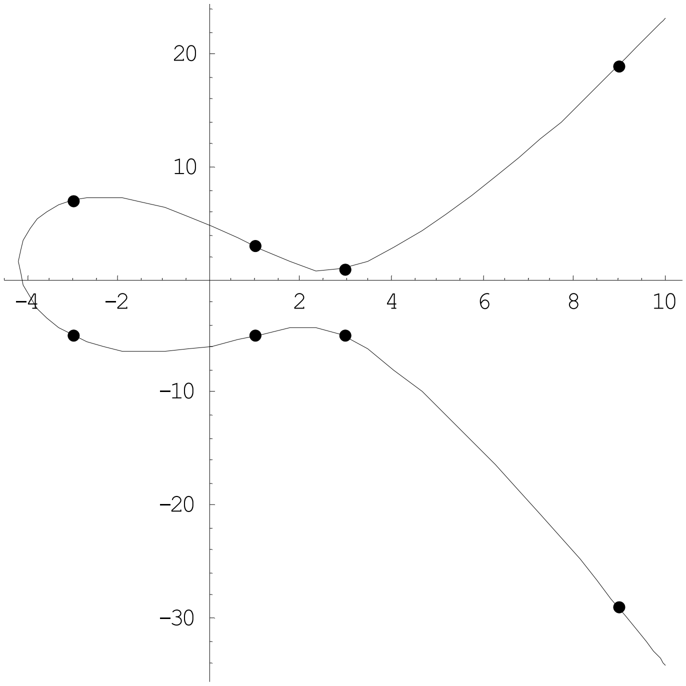


$\boldsymbol{Z_9}$

Here's the case where $y^2 + xy + y = x^3 - x^2 - 14x + 29$.

Here we have that $E_{\text{tors}} = Z_9 = \langle(3,1)\rangle = \{(3,1), (-3,7), (1,-5), (9,-29), (9,19), (1,3), (-3,-5), (3,-5), O\}$.

Here's what it looks like:

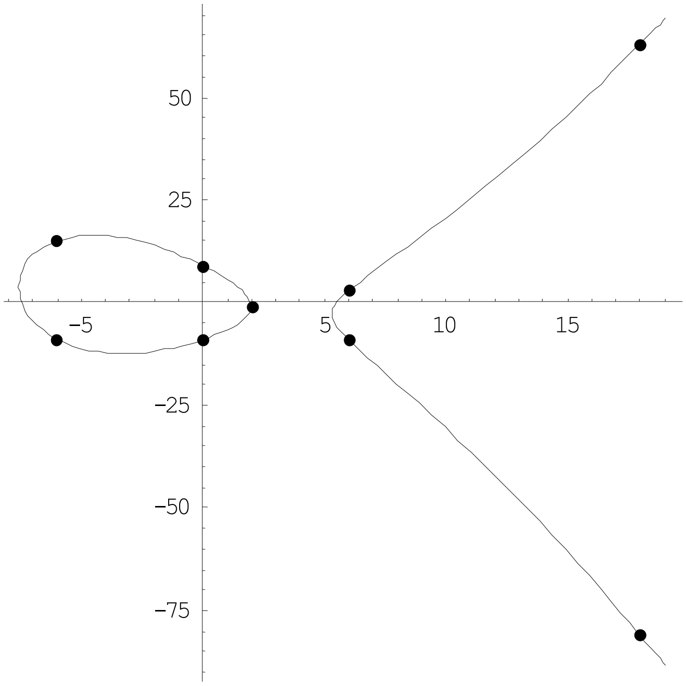


Z_{10}

Here's the case where $y^2 + xy = x^3 - 45x + 81$.

Here we have that $E_{\text{tors}} = Z_{10} = \langle(0,9)\rangle = \{(0,9), (6,3), (-6,-9), (18,-81), (2,-1), (18,63), (-6,15), (6,-9), (0,-9), O\}$.

Here's what it looks like:

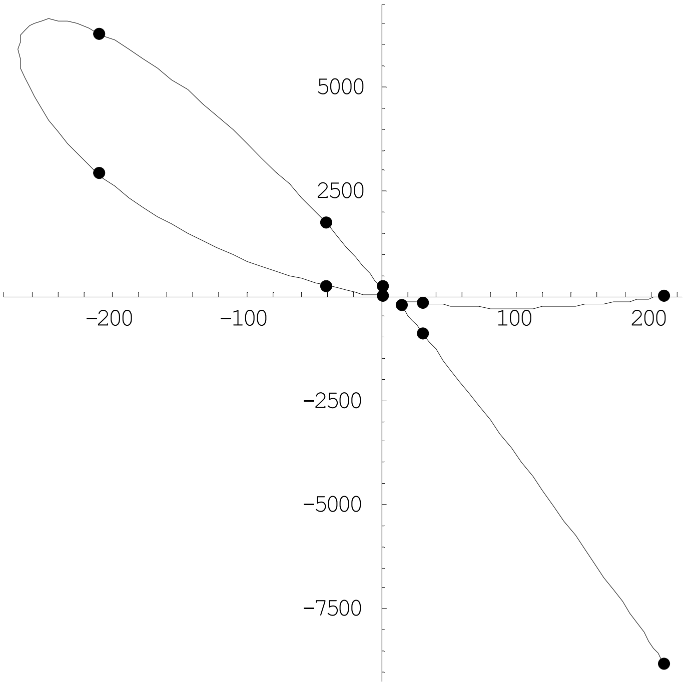


$\boldsymbol{Z}_{12}$

Here's the case where $y^2 + 43xy - 210y = x^3 - 210x^2$.

Here we have that $E_{\text{tors}} = Z_{12} = \langle(0,210)\rangle = \{(0,210), (210,0), (-42,1764), (30,-180), (-210,6300), (14,-196), (-210,2940), (30,-900), (-42,250), (210,-8820), (0,0), O\}$.

Here's what it looks like:

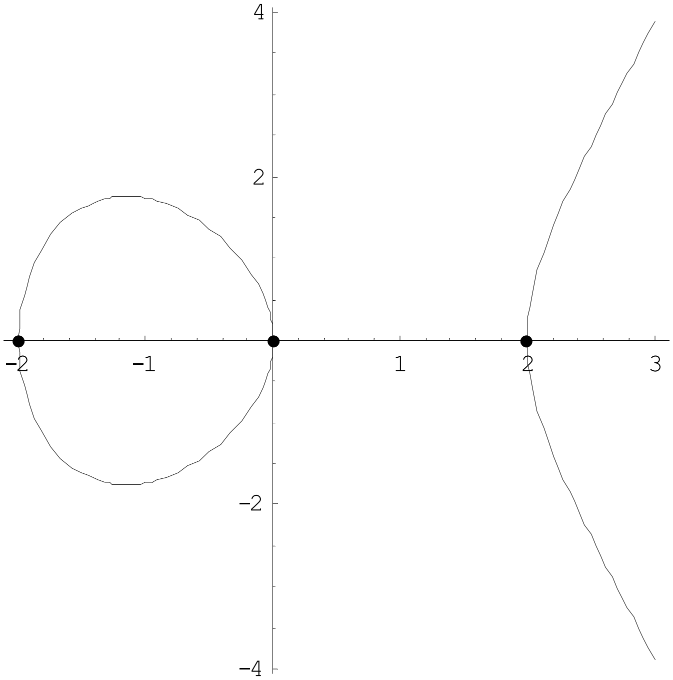


Z_2 x Z_2

Here's the case where $y^2 = x^3 - 4x$.

Here we have that $E_{\text{tors}} = Z_2 \times Z_2 = \langle(2,0),(0,0)\rangle = \{(2,0), (0,0), (-2,0), O\}$.

Here's what it looks like:

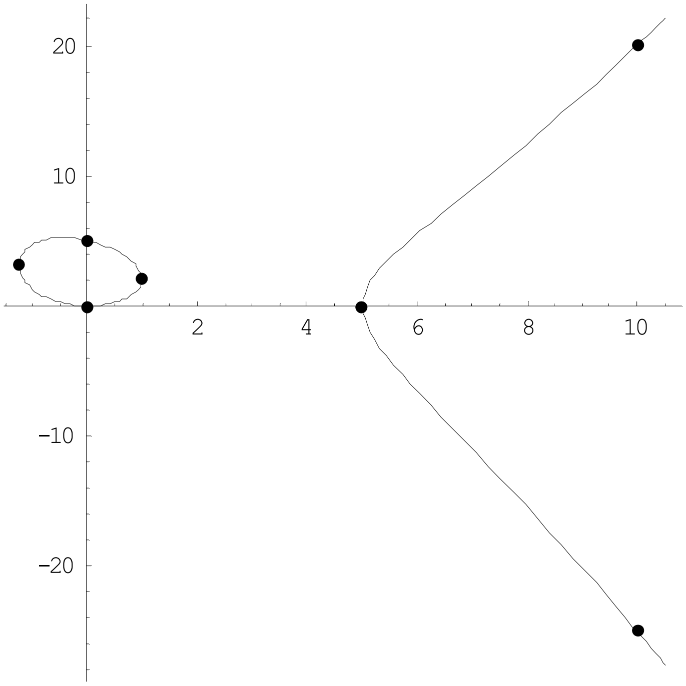


Z_2 x Z_4

Here's the case where $y^2 + xy - 5y = x^3 - 5x^2$.

Here we have that $E_{\text{tors}} = Z_2 \times Z_4$ = <(10,20),(1,2)> = {(1,2), (10,-25), (0,5), (0,0), (-5/4,25/8), (5,0), (10,20), O}.

Here's what it looks like:

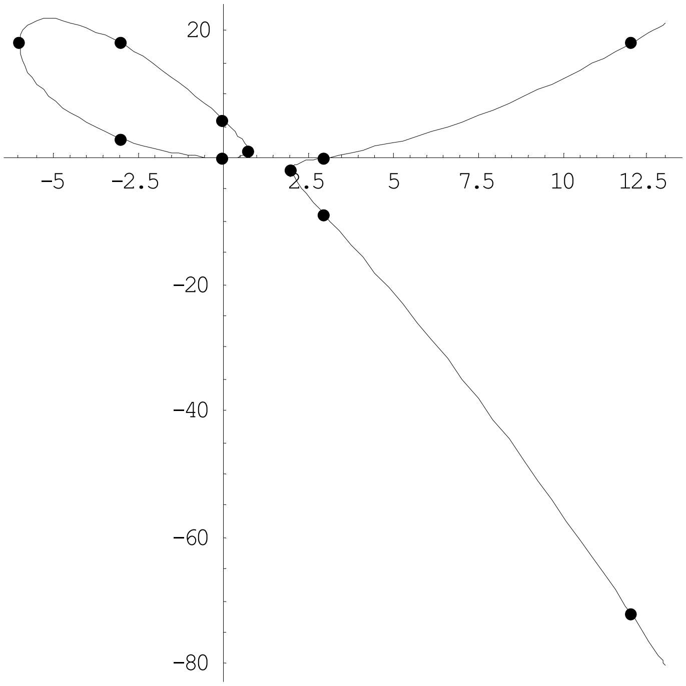


Z_2 x Z_6

Here's the case where $y^2 + 5xy - 6y = x^3 - 3x^2$.

Here we have that $E_{\text{tors}} = Z_2 \times Z_6 = \langle(-3,18),(2,-2)\rangle = \{(-6,18), (12,-72), (12,18), (3/4,9/8), (-3,18), (-3,3), (0,0), (3,0), (2,-2), (0,6), (3,-9), O\}$.

Here's what it looks like:



$Z_2 \times Z_8$

Here's the case where $y^2 + 17xy - 120y = x^3 - 60x^2$.

Here we have that $E_{\text{tors}} = Z_2 \times Z_8 = \langle(30,-90), (-40,400)\rangle = \{(30,-90), (60,-900), (240,1800), (24,-144), (240,-5760), (60,0), (30,-300), (0,0), (-12, 288), (-30,180), (15/4,255/8), (-30,450), (-12,36), (0,120), (-40,400), O\}$.

Here's what it looks like: